Complete Convergence Theorem for the Contact Process on $T^d \times Z$

[1]

Abstract

We consider the contact process on a branching plane $T_d \times Z$, which is the product of a regular tree T and the line Z. We show when $d \geq 9$ that there are two phase transitions as the infection parameter λ is varied. And above the second critical point, we have the complete convergence theory hold. We also show that at the first critical point the contact process on $T_d \times Z$ dies out.

Key words. Contact process, $T_d \times Z$, complete convergence theory

1. Introduction

In this paper we shall consider the contact process with infection parameter λ on the branching plane $T_d \times Z$. Before 1990, the contact process was mostly studied on the $d-$dimensional cubic lattice Z^d. And ergodic behavior of the contact process on Z^d was completely characterized in the fundamental paper by Bezuidenhout and Grimmett (1990). Interest in the behavior of the contact process on trees was raised by Pemantle (1992). It was shown that the contact process on homogeneous trees has at least two different critical points and between them the system can survive in a global sense, but not in a local sense. From that paper on, a lot of paper on the problem on trees appears. In particular, Zhang Yu and Salzano & Schonmann used two different method solved complete convergence theory on trees. But we don't know if the complete convergence theory is true on Cayley graphs. This is the motivation for us to study the contact process on $T_d \times Z$, for it is a Cayley graph but different from T_d and Z^d.

We will denote by $(\xi^{\eta}_{G,t} : t \geq 0)$ the contact process on the graph G starting from the configuration η. When G is the whole branching plane $T_d \times Z$, G will be omitted

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in this notation. Without confusion, we shall also use G to denote the vertices set of the graph G. A vertex in $T_d \times Z$ is denoted as $\mathbf{x} = (x_1, x_2),\ x_1 \in T_d, x_2 \in Z$. We write (x_1, x_2) instead of $\mathbf{x}$ when it is necessary.

First we shall make some introduction to the standard graphical construction of the contact process. We associate each site $\mathbf{x} \in T_d \times Z$ with $d+4$ independent Poisson processes, one with rate 1, and $d+3$ others with rate λ. Let $\{T_n^{\mathbf{x},k} : n \geq 1\}, k = 0, 1, 2, \cdots$, d+3 be the arrival times of these $d+4$ processes, respectively; the process $\{T_n^{\mathbf{x},0} :\ n \geq 1\}$ has rate 1, the others has rate λ. For each $\mathbf{x}$ and $n \geq 1$ we write a δ mark at the point $(\mathbf{x}, T_n^{\mathbf{x},0})$ to indicate a death will occur if $\mathbf{x}$ is occupied; while if $k \geq 1$ we draw arrows from $(\mathbf{x}, T_n^{\mathbf{x},k})$ to $(\mathbf{x}^{(k)}, T_n^{\mathbf{x},k})$ to indicate if $\mathbf{x}$ is occupied then there will be a birth from $\mathbf{x}$ to $\mathbf{x}^{(k)}$, where $\mathbf{x}^{(k)}, k = 1, 2, \cdots d+3$ are the neighbors of $\mathbf{x}$ arranged in some arbitrary order. We say that there is a path from $(\mathbf{x}, s)$ to $(\mathbf{y}, t)$ if there is a sequence of times $s = s_0 < s_1 \cdots < s_n < s_{n+1} = t$ and spatial locations $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_n = \mathbf{y}$ so that for $i = 1, 2, \cdots, n$ there is an arrow from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ at time s_i and the vertical segments $\{\mathbf{x}_i\} \times (s_i, s_{i+1})$ for $i = 0, 1, \cdots n$ do not contain any δ. We will say that the path is inside a set $\mathbf{D} \subset T_d \times Z$ case all $\mathbf{x}_i$ are contained in $\mathbf{D}$.

Given a tree T_d, we immerse the graph Z^+ which has vertices $\{0, 1, 2, \cdots\}$ and edges connecting point which differ by one unit into it. The root of T_d is denoted as 0. The vertex i in Z^+ is an arbitrary neighbor of $i-1$, and different from $\{0, \cdots, i-1\}$. An important subgraph of T_d^+ is obtained from this tree by removing one of the neighbors of the root and defining it as the remaining connected component which contains the root. We will suppose that the removed vertex is not the vertex 1, so that the set of sites $\{0, 1, 2, \cdots\}$ is contained in the set of vertices of the new graph. This new graph will be denoted as T_d^+. For any $v \in T_d^+$, if we remove the vertex adjacent to x and stays nearer to the root than v, T_d is broken into two parts; we denote the connected part which contains v as $T_d^+(v)$. The root of $T_d \times Z$ will be denoted as $(0, 0)$.

We define the distance between sites in a graph G by the number of edges of the shortest path between them. Given a tree T_d, let $B(x,N) = \{v \in T_d : |v - x| \leq N\}$. ($|v - x|$ is the distance between v and x, and we shall use $|v|$ to denote the distance between v and the root 0.) We use the abbreviation $B(N) = B(0,N)$. For a branching plane $T_d \times Z$, we use $\mathbf{B}(x,N)$ to denote $B(x,N) \times Z$, correspondingly $\mathbf{B}(N) = B(N) \times Z$.

For any $\mathbf{S} \subset T_d \times Z$, we define

$$f(\mathbf{S}) = \{x \in T_d : \text{ there is some } j \in Z \text{ which has } (x,j) \subset \mathbf{S}\}.$$

Define a branch of a vertex $x \in T_d$ to be the connected component of the subgraph obtained by removing a subset of d edges, each having an endpoint at x, so there are $d+1$ distinguished branches of x, we denote them as $\beta_1^x, \beta_2^x, \cdots, \beta_{d+1}^x$. The $d+1$ branches of the root 0 will be denoted as $\beta_1, \beta_2, \cdots, \beta_{d+1}$. For any $v \in T_d^+$, we let $\beta_i^v \in T_d^+, i = 1, \cdots d$. So we have $\beta_{d+1} \cup T_d^+ = T_d$. Given a vertex $\mathbf{x} \in T_d \times Z, f(\mathbf{x}) = x$. Define $\mathcal{B}_i^{\mathbf{x}} = \beta_i^x \times Z$ $(i = 1, \cdots, d+1)$. We call $\mathcal{B}_1^{\mathbf{x}}, \mathcal{B}_2^{\mathbf{x}}, \cdots, \mathcal{B}_{d+1}^{\mathbf{x}}$ are the $d+1$ branches of $\mathbf{x} \in T_d \times Z$. Correspondingly $\{\mathcal{B}_i\}_{i=1 \cdots d+1}$ are the $d+1$ branches of the root $(0,0)$ in $T_d \times Z$.

Given $S \subset T_d$, we say a site $x \in S$ is in the border of S, if at least one of the $d+1$ branches emanating from x has no site S(except x). Denote the border of S by $B(S)$. We say $x \in B_o(S)$, if at least one of the $d+1$ branches emanating from x which has no site in S is in T_d^+. It is well known

$$B(S) \geq (1 - \frac{1}{d})|S|. \qquad (*)$$

So

$$B_o(S) \geq (1 - \frac{1}{d})|S| - 1$$

For a proof of (*) see Lemma 6.2 in Pemantle(1992). Let $\mathbf{S}$ be a subset of $T_d \times Z$. We denote $\mathbf{B}_o(\mathbf{S}) = \mathbf{S} \cap f(\mathbf{S}) \times Z$. By the definition of $B_o(S)$, for any $v \in B_o(S)$ we can choose a branch of v, which is in T_d^+ and has no site in S except v. We denote it as $\beta_{g(v)}^v$.

Order the vertices in $B(k,k)$, and let all the vertices in $T_d^+(k)$ is before β_{d+1}^k. Define

$$h(\xi_{\mathbf{B}(k,k),t}^{(k,0)}) = \xi_{\mathbf{B}(k,k),t}^{(k,0)} \cap \{v_t\} \times Z,$$

where v_t is the first vertex in $\xi_{\mathbf{B}(k,k),t}^{(k,0)}$ which satisfies with

$$|\xi_{\mathbf{B}(k,k),t}^{(k,0)} \cap \{v\} \times Z| = \max_{x \in B(k,k)} \{|\xi_{\mathbf{B}(k,k),t}^{(k,0)} \cap \{x\} \times Z|\}.$$

Lemma 1. On $\Omega_\infty = \{|\xi_t^{(0,0)}| \geq 1,\ \forall t > 0\}$ we have almost surely that

$$\lim_{t\to\infty} |\xi_t^{(0,0)}| \longrightarrow \infty.$$

Lemma 2. If $P(\Omega_\infty) > 0$, then we have $\limsup_{t\to\infty} E|f(\xi_{\mathcal{B}_1,t}^{(0,0)})| = \infty$.

*P*roof. According to Lemma 1, we know that

$$P(\lim_{t\to\infty} |\xi_t^{(0,0)}| = \infty) = P(\Omega_\infty).$$

So for all $M > 0$ we can choose t_1 and $k = k(t_1)$, which satisfies with

$$P(|\xi_{\mathbf{B}(k,k),t}^{(k,0)}| \geq M^2) \geq \frac{1}{2} P(\Omega_\infty).$$

Then

$$\begin{aligned}
P(|\xi_{\mathbf{B}(k,k),t}^{(k,0)}| \geq M^2) &\leq P(|f(\xi_{\mathbf{B}(k,k),t}^{(k,0)}) h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \geq M^2) \\
&\leq P(|f(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \leq M) + P(|h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \geq M) \\
&\leq P(|h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \geq M, f(h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})) \in T_d^+(k)) \\
&\quad + P(|h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \geq M, f(h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})) \in \beta_{d+1}^k) \\
&\quad + P(|f(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \leq M) \\
&\leq 2P(|h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \geq M, f(h(\xi_{\mathbf{B}(k,k),t}^{(k,0)})) \in T_d^+(k)) \\
&\quad + P(|f(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| \leq M)
\end{aligned}$$

So we have

$$P(|f(\xi_{\mathbf{B}(k,k),t}^{(k,0)})| > M) \geq \frac{1}{6} P(\Omega_\infty) \tag{3.1}$$

or

$$P(|h(\xi^{(k,0)}_{\mathbf{B}(k,k),t})| > M, f(h(\xi^{(k,0)}_{\mathbf{B}(k,k),t})) \in T_d^+(k)) \geq \frac{1}{6}P(\Omega_\infty). \tag{3.2}$$

For any sunset $\mathbf{S} \subset T_d \times Z$, we have

$$\begin{aligned} E|f(\xi^{\mathbf{S}}_{\mathcal{B}_1,t})| &\geq E|f(\xi^{\mathbf{B}_o(\mathbf{S})}_{\mathcal{B}_1,t})| \\ &\geq \sum_{x \in B_o(f(\mathbf{S}))} E|f(\xi^{\mathbf{B}_o(\mathbf{S})}_{\mathcal{B}_1,t}) \cap \beta^x_{g(x)}|) \\ &\geq \sum_{x \in B_o(f(\mathbf{S}))} E|f(\xi^{(x,0)}_{\mathcal{B}_1,t}) \cap \beta^x_{g(x)}|) \end{aligned}$$

Given $\mathbf{x}, \mathbf{v} \in \mathcal{B}_1,\ |\mathbf{x}| \geq |\mathbf{v}|$, using the method of coupling we can easily get

$$E|f(\xi^{\mathbf{x}}_{\mathcal{B}_1,t})| \geq E|f(\xi^{\mathbf{v}}_{\mathcal{B}_1,t})| \tag{3.4}$$

Now if (3.1) is satisfied, using (3.3) and (3.4) and the symmetry of the tree, we get

$$\begin{aligned} E|f(\xi^{(k,0)}_{\mathcal{B}_1,lt_1})| &= E(E(|f(\xi^{(k,0)}_{\mathcal{B}_1,lt_1})||\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})) \\ &\geq E \sum_{x \in B_o(f(\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})) \cap T_d^+(k)} E(|f(\xi^{(x,0)}_{\mathcal{B}_1,(l-1)t_1})) \cap \beta^x_{g(x)}|) \\ &\geq E(|f(\xi^{(k,0)}_{\mathcal{B}_1,(l-1)t_1}) \cap \beta_1|) E \sum 1_{x \in B_o(f(\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})) \cap T_d^+(k)} \\ &\geq \frac{1}{d+1} E(|f(\xi^{(k,0)}_{\mathcal{B}_1,(l-1)t_1})|)(\frac{d}{d+1}((1-\frac{1}{d})E|f(\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})| - 1)) \\ &\geq (\frac{d}{(d+1)^2}(\frac{M}{6}(1-1/d)P(\Omega_\infty) - 1))^l \end{aligned}$$

Then

$$E|f(\xi^{(0,0)}_{\mathcal{B}_1,lt_1+k})| \geq p_k(\frac{d}{(d+1)^2}(\frac{M}{6}(1-1/d)P(\Omega_\infty) - 1))^l, \tag{3.5}$$

where p_k is the probability that there is a path from $((0,0),0)$ to $((k,0),k)$ inside $\mathcal{B}_1$.

If (3.2) is satisfied, we have

$$\begin{aligned} E|f(\xi^{(k,0)}_{\mathcal{B}_1,l(t_1+M)})| &= E(E(|f(\xi^{(k,0)}_{\mathcal{B}_1,l(t_1+M)})||\xi^{(k,0)}_{\mathbf{B}(k,k),t_1}) \\ &\geq E(E(|f(\xi^{h(\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})}_{\mathcal{B}_1,l(t_1+M)-t_1} 1_{f(h(\xi^{(k,0)}_{\mathbf{B}(k,k),t_1})) \in T_d^+(k)})|) \end{aligned}$$

Given $\mathbf{A} = \{v\} \times B$, where $|v| > k$ and B is a subset of Z, we have

$$\begin{aligned}
E|f(\xi^{\mathbf{A}}_{\mathcal{B}_1,l(t_1+M)-t_1})| &= E(E(|f(\xi^{\mathbf{A}}_{\mathcal{B}_1,l(t_1+M)-t_1})||\xi^{\mathbf{A}}_{\mathcal{B}_1,M})) \\
&\geq E(\sum_{x\in B_o(f(\xi^{\mathbf{A}}_{\mathcal{B}_1,M}))} E(|f(\xi^{(x,0)}_{\mathcal{B}_1,(l-1)(t_1+M)}) \cap \beta^x_{g(x)}|) \\
&\geq E(\sum_{x\in B_o(f(\xi^{\mathbf{A}}_{\mathcal{B}_1,M}))\cap\partial B^+(v,M)} E(|f(\xi^{(x,0)}_{\mathcal{B}_1,(l-1)(t_1+M)}) \cap \beta^x_{g(x)}|)) \\
&\geq E(|f(\xi^{(k,0)}_{\mathcal{B}_1,(l-1)(t_1+M)}) \cap \beta^k_1|)(d^M(1-(1-a^M)^{|A|}) \\
&= \frac{1}{d+1}E(|f(\xi^{(k,0)}_{\mathcal{B}_1,(l-1)(t_1+M)})|)(d^M(1-(1-a^M)^{|A|}),
\end{aligned}$$

here $\partial B^+(v,M) = \{x \in T_d^+ : |x-v| = M, |x| > |v|\}$ and a is the probability that for the contact process on $T_d \times Z$ staring from $(0,0)$, $(1,0)$ is infected at time 1, and no other particles was infected before time 1. So

$$E|f(\xi^{(k,0)}_{\mathcal{B}_1,l(t_1+M)})| \geq (\frac{d^M}{d+1}(1-(1-a^M)^M)\frac{1}{6}P(\Omega_\infty))^l$$

$$E|f(\xi^{(0,0)}_{\mathcal{B}_1,l(t_1+M)+k})| \geq p_k(\frac{d^M}{d+1}(1-(1-a^M)^M)\frac{1}{6}P(\Omega_\infty))^l \tag{3.6}$$

Clearly,

$$\lim_{t\to\infty} 1-(1-a^M)^M = 1-e^{-1}.$$

So we can choose M large enough such that

$$\frac{d-1}{(d+1)^2}(\frac{M}{6}P(\Omega_\infty)-1) \geq 2 \tag{3.7}$$

and

$$\frac{d^M}{d+1}(1-(1-a^M)^M)\frac{1}{6}P(\Omega_\infty) \geq 2. \tag{3.8}$$

Now using $(3.5),(3.6),(3.7))$ and (3.8), for any C, we can choose l, i.e. T, large enough such that

$$E(|f(\xi^{(0,0)}_{\mathcal{B}_1,T})|) \geq C.$$

So we have $\limsup_{t\to\infty} E(|f(\xi^{(0,0)}_{\mathcal{B}_1,t})|) = \infty$.

Theorem 1. When $\lambda = \lambda_1$, we have $P_{\lambda_1}(\Omega_\infty) = 0$, i.e. the contact process dies out at the first critical point.

Proof. If $P_{\lambda_1}(\Omega_\infty) > 0$, we have $\limsup_{t\to\infty} E_\lambda |f(\xi^{(0,0)}_{\mathcal{B}_1,t})| = \infty$. So for any $C > 0$, we can choose T large enough such that $E|f(\xi^{(0,0)}_{\mathcal{B}_1,T})| > 2C$. And then we can find $\lambda < \lambda_1$, which satisfies have $E_{\lambda_1} f(\xi^{(0,0)}_{\mathcal{B}_1,t}) > C$. For this λ, we define a new process $\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,t}$ as follows. $\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,t}$ run as $\xi^{(0,0)}_{\mathcal{B}_1,t}$ up to time T. At time T, we suppress all the particles which are not in $\mathbf{B}_o(\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,T})$. For any $v \in B_o(f(\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,T}))$, we choose randomly one infected particle from $\{v\} \times Z \cap \tilde{\xi}^{(k,0)}_{\mathcal{B}_1,T}$ and suppress the others. And we restrict the spatial evolution of the remaining particles in the following way. Each preserved particles at time T generates a process for which births are allowed only on the branch, adjacent to the initial particles, and which was empty at time T. We create like this at time T, $|B_o(f(\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,T}))|$ contact processes which are independent of the other. We repeat the proceed steps at all times kT: We suppress all the particles which are not in $\mathbf{B}_o(\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,kT})$. For any $v \in B_o(f(\tilde{\xi}^{(0,0)}_{\mathcal{B}_1,kT}))$, we choose randomly one infected particle from $\{v\} \times Z \cap \tilde{\xi}^{(k,0)}_{\mathcal{B}_1,kT}$ and suppress the others. We also restrict the spatial evolution of the remaining particles in the following way. Each preserved particles at time kT generates a process for which births are allowed only on the branch of $T_d \times Z$, which is adjacent to the initial particles, and which is empty at time kT. Between times kT, and $(k+1)T$, the process evolves using the graphical construction. If we define the discrete-time process $b_k = |\tilde{\xi}_{\mathcal{B}_1,kT}|$ then b_k is a Galton-Watson process. We choose C large enough such that $(1-1/d)/M - 1 > 1$, then $Eb_1 = |B_o(f(\xi^{(0,0)}_{\mathcal{B}_1,T}))| \geq (1-\frac{1}{d})C - 1 > 1$. So $P(b_k \geq 1, \text{ for all } k > 0) > 0$. Hence $P(|\xi^{(0,0)}_{\mathcal{B}_1,t}| \geq 1 \text{ for all } t > 0) > 0$, which is conflict with the definition of λ_1, so $P_{\lambda_1}(\Omega_\infty) = 0$.

Theorem 2. When $d \geq 9$, we have $\lambda_1 < \lambda_2$.

Proof. First we define the weight of a vertex $\mathbf{v} \in T_d \times Z$ to be $w(\mathbf{v}) = (2(d+1))^{-n/2}$, where n is the distance from the root $(0,0)$ to $\mathbf{v}$. Let

$$W(t) = \sum_{\mathbf{v}} w(\mathbf{v}) 1_{\mathbf{v} \in \xi^{(0,0)}_t} = \text{ total weight at time } t.$$

Then

$$
\begin{aligned}
E(W(t+dt)|\xi_t^{(0,0)}) &= E(\sum_{\mathbf{v}} 1_{\mathbf{v}\in\xi_{t+dt}^{(0,0)}} w(\mathbf{v})|\xi_t^{(0,0)}) \\
&\le W(t) + \sum_{\mathbf{v}} 1_{\mathbf{v}\in\xi_t^{(0,0)}} w(\mathbf{v})(2\lambda\sqrt{2(d+1)} + (d+1)\lambda\sqrt{2(d+1)} - 1) \\
&= W(t) + dtW(t)(2\sqrt{2(d+1)}\lambda - 1).
\end{aligned}
$$

So for $\lambda < 2\sqrt{2(d+1)}$, $EW(t)$ goes to 0 exponentially fast, hence so does the contact process is not locally survive. Then

$$\lambda_2 \ge \frac{1}{\sqrt{2(d+1)}}.$$

By Pemantle, we know for the contact process on T_d,

$$\lambda_1(T_d) \le 1/(d-1).$$

So when $d \ge 9$, we have $\lambda_1 < \lambda_2$.

Complete Convergence Theorem

Define

$$u_n = P(\xi_t^{(0,0)}((n,0)) = 1 \text{ for some } t \ge 0)$$

From the inequality $u_{n+m} \ge u_n u_m$, it follows that $\lim_{n\to\infty}(u_n)^{1/n} = \alpha = \alpha(\lambda)$ exists.

Define

$$Y_{n,s} = P(\text{ There is a path from } ((0,0),0) \text{ to } ((n,0),s) \text{ inside } T_d^+ \times Z).$$

Lemma 3. There exists a sequence $(s(n))_{n\ge1}$ such that

$$\lim_{n\to\infty}(Y_{n,s(n)})^{1/n} = \alpha.$$

Proof. It is clearly that for any sequence $s(n)$,

$$\limsup_{n\to\infty}(Y_{n,s(n)})^{1/n} \le \limsup_{n\to\infty}(u_n)^{1/n} = \alpha.$$

Define

$$\begin{aligned} V_{m,k} &= P(\text{There is a path from } ((0,0),0) \text{to } \; ((m,0),t) \text{ for some } t \leq k \\ &\qquad \text{inside } \mathbf{B}(k)) \\ &= P(\xi^{(0,0)}_{\mathbf{B}(k);t}((m,0)) = 1 \text{ for some } t \leq k). \end{aligned}$$

Clearly

$$\lim_{k\to\infty} V_{m,k} = u_m. \tag{1.1}$$

Next define

$$\begin{aligned} W_{n,k} &= P(\text{There is a path from } ((0,0),0) \text{to } ((n,0),t) \text{ for some } t \leq kn \\ &\qquad \text{inside } T_d^+ \times Z) \\ &= P(\xi^{(0,0)}_{T_d^+ \times Z;t}((n,0)) = 1 \text{ for some } t \leq kn). \end{aligned}$$

We will argue next

$$W_{n,k} \geq C_{m,k}(V_{m,k})^{\lceil n/m \rceil - I}, \tag{1.2}$$

where $C_{m,k}$ is a positive quantity which does not depend on n. To prove this, we set $I = \min\{i \in Z^+ : im > k\} = \min\{i \in Z^+ : B(im,k) \subset T_d^+)$. Consider now the sequence of sites $x_1 = Im, x_2 = (I+1)m, \cdots, x_J = \lceil n/m \rceil m$. Define τ_i as the first time that that there is a path from x_{i-1} to x_i after τ_{i-1}; Let $\tau_0 = 1$. Clearly, the random times $\{\tau_i : \; i \geq 1\}$ are stopping times. Suppose that all of the following events occur: (A) There is a path go from $((0,0),0)$ to $((x_1,0),1)$ without exiting $T_d^+ \times Z$. (B) For each $i = 0, \cdots, J-1$, there is a path from x_i to x_{i+1} beginning at time τ_i and ending at time τ_{i+1} that does not exit $\mathbf{B}_k(x_i)$, and $|\tau_{i+1} - \tau_i| \leq k$. Since the times τ_i are stopping times and the event (A) (B) involving non-overlapping parts of the percolation structure, it follows by the translation invariance of the graphical construct and the strong Markov property of the underlying Poisson processes that the probability of the existence of a path from the root $(0,0)$ to $(x_J,0)$ is at least

$C_{m,k}(V_{m,k})^{\lceil n/m\rceil - I}$. Hence

$$\begin{aligned} W_{n,k} &= P(\xi^{(0,0)}_{T_d^+\times Z;t}((n,0)) = 1 \text{ for some } t \le kn), \\ &\ge P(\xi^{(0,0)}_{T_d^+\times Z;t}((\lceil n/m\rceil m,0)) = 1 \text{ for some } t \le kn), \\ &\ge C_{m,k}(V_{m,k})^{\lceil n/m\rceil - I} \end{aligned}$$

Define

$$\begin{aligned} \overline{W}_{n,k} &= \max_{j=0,1,\cdots,nk-1} P(\text{There is a path from } ((0,0),0) \text{ to } ((n,0),t) \\ &\qquad \text{inside } T_d^+ \times Z \text{ for some } t \in [j,j+1]) \\ &= \max_{j=0,1,\cdots,nk-1} P(\xi^{(0,0)}_{T_d^+\times Z;t}((n,0)) = 1 \text{ for some } t \in [j,j+1]). \end{aligned}$$

The event in this definition and the event that there is no death mark in $\{(n,0)\} \times [j,j+1]$ are both increasing, therefore, for a proper choice of $s(n)$,

$$Y_{n,s(n)} \ge e^{-1}\overline{W}_{n,k} \ge \frac{1}{kn} e^{-1} W_{n,k}. \tag{1.3}$$

Using now $(1.3), (1.2), (1.1)$ and the definition of α,

$$\begin{aligned} \liminf_{n\to\infty}(Y_{n,s(n)})^{1/n} &= \lim_{m\to\infty}\lim_{k\to\infty}\liminf_{n\to\infty}(Y_{n,s(n)})^{1/n} \\ &\ge \liminf_{m\to\infty}\liminf_{k\to\infty}(V_{m,k})^{1/m} \\ &= \liminf_{m\to\infty}(u_m)^{1/m} \\ &= \alpha \end{aligned}$$

finishing the proof.

Lemma 4. If $\alpha(\lambda) > 1/\sqrt{d}$, then $\inf_{t\ge 0} P(\xi^{(0,0)}_{T_d^+\times Z;t}((0,0)) = 1) > 0$.

Proof. Suppose that $\alpha(\lambda) > 1/\sqrt{d}$. Then according to Lemma 1, we can take n and s, such that

$$(Y_{n,s})^{1/n} = a > 1/\sqrt{d}. \tag{2.1}$$

We will show that for a proper choice of a positive integer l,

$$\inf_{i=0,1,2,\cdots} P(\xi_{T_d^+\times Z;2ils}((0,0)) = 1) > 0 \tag{2.2}$$

This clearly suffices for our purposes, since then

$$\inf_{t\geq 0} P(\xi^{(0,0)}_{T_d^+\times Z;t}((0,0))=1) \geq e^{-2ls} \inf_{i=0,1,2,\cdots} P(\xi^{(0,0)}_{T_d^+\times Z;2ils}((0,0))=1) > 0.$$

In order to prove (2.1), we consider now the following modification of the contact process on $T_d^+\times Z$. Untill time s, we run the usual contact process on $T_d^+\times Z$, started from a single particle at the root. At time s, we remove all particles except for those which are in $(B(n)\setminus B(n-1))\times\{0\}$; from this time on, we keep the set $\mathbf{B}(n-1)$ free of particle, but untill time $2s$ we let the system evolve in the remaining vertices with the usual contact process rules. At time $2s$ we remove all particles except for those which are in $(B(2n)\setminus B(2n-1))\times\{0\}$; from this time on, we keep the set $\mathbf{B}(2n-1)$ free of particles, but untill time $3s$ we let the system evolve in the remaining vertices with the usual contact process rules. The modification should now be clear. For each j, at time js, we remove all particles except for those which are in $(B(jn)\setminus B(jn-1))\times\{0\}$; from time js on, we keep the set $\mathbf{B}(jn-1)$ free of particles, but untill time $(j+1)s$, we let the system evolve in the remaining vertices with the usual contact process rules.

If we only pay our attention on this modified process at time js, and define the modified process at time js to be a process $\{Z_j\}_{j=0,1,2,\cdots}$. It is clearly that all of the particles in Z_j are in $(B(jn)\setminus B(jn-1))\times\{0\}$). Let $|Z_j|$ be the number of particles in this modified process at time js . We can see that $(|Z_j|)_{j=0,1,2,\cdots}$ is a branching process with mean offspring number $d^nY_{n,s}=(da)^n$. Since the offspring distribution has a finite support (namely $\{0,1,\cdots,d^n\}$), and in particular a finite second moment, it follows from standard branching process theory that for some random variable X with mean $EX=1$,

$$\frac{|Z_j|}{(da)^{jn}} \longrightarrow X \qquad a.s. \text{ as } j\longrightarrow\infty.$$

In particular there is $\epsilon>0$ such that

$$P(|Z_l|\geq \frac{(da)^{nl}}{2}) \geq \epsilon, \tag{2.3}$$

for all large enough l.

Choose l large enough for (2.2) to hold, and also so that

$$(1-\frac{\epsilon}{2}a^{nl})^{(da)^{nl/2}} \leq \frac{1}{2}. \tag{2.4}$$

This last requirement can be fulfilled because

$$\begin{aligned}\lim_{l\to\infty}(1-\frac{\epsilon}{2}a^{nl})^{(da)^{nl/2}} &\leq \lim_{l\to\infty}\exp(-\frac{\epsilon}{2}a^{nl}\frac{(da)^{nl}}{2}) \\ &= \lim_{l\to\infty}\exp(-\frac{\epsilon}{4}(da^2)^{nl}) = 0,\end{aligned}$$

since $da^2 > 1$, by (2.1).

Now define

$$r_i = P(\xi^{(0,0)}_{T_d^+\times Z;2ils}((0,0)) = 1).$$

We shall show inductively in i that

$$r_i \geq \frac{\epsilon}{2}a^{nl}, \tag{2.5}$$

verifying therefore the validity of (2.2). For $i = 0$ inequality (2.4) is clearly true, and we will show that if it is true for i, it is true for $i+1$. From the Markov property and attractiveness of the contact process we have

$$\begin{aligned} r_{i+1} &= P(\xi_{T_d^+\times Z;2(i+1)ls}(0,0) = 1) \\ &\geq P(\xi_{T_d^+\times Z;(2i+1)ls}(x) = 1 \text{ for some } x \in B(nl)\setminus B(nl-1))Y_{ln,ls} \\ &\geq P(\xi_{T_d\times Z;(2i+1)ls}(x) = 1 \text{ for some } x \in B(nl)\setminus B(nl-1))(Y_{n,s})^l. \end{aligned}$$

It is clear that the contact process on $T_d^+ \times Z$ dominates the modified process introduced above, immediately before the definition of $|Z_j|$, in the usual sense that it has a particle at any space-time location in which this other one has a particle. The same is true if we consider a somewhat different modification of the contact process on $T_d^+ \times Z$, in which until time ls we make the same modifications as before, but from this time on we keep only the sites in $B(nl-1)$ free of particles, and we let the system evolve in the remaining vertices with the usual contact process rules. Using these observations, the last display above yields

$$r_{i+1} \geq P(Z_l > \frac{(da)^{nl}}{2}) \cdot (1 - (1 - r_i)^{\frac{(da)^{nl}}{2}}) \cdot (Y_{n,s})^l \geq \frac{\epsilon}{2} a^{nl},$$

where in the second inequality we are using (2.3), the induction hypothesis (2.5), (2.4) and (2.1)

Lemma 5.(Salzano and Schonmann) For each infinite connected graph G and each value $\lambda > 0$, the following statements are true.
(a) The property $r\&pc$ are equal to $\lim_{N \to \infty} \liminf_{t \to \infty} P(\xi_t^{R(0,N)} \cap R(0,N) \neq \phi) = 1$, ($R(0,N)$ is the ball of center 0 and radius N in $T_d \times Z$.)
(b)Complete Convergence Theorem is equivalent to having simultaneously pc and $r = s$.

Lemma 6.(Salzano and Schonmann)Suppose that for a graph G, there exist a $\delta > 0$, so that for every $x \in G$, $P(o \in \xi_t^x$ for some $t \geq 0) \geq \delta > 0$, Then $r = s$ hold.

Theorem 3. When $\lambda > \lambda_2$ we have the complete convergence theorem hold, i.e., for any $\mathbf{A} \in T_d \times Z$,

$$\xi_t^{\mathbf{A}} \Rightarrow P(\tau^{\mathbf{A}} < \infty)\delta_0 + P(\tau^{\mathbf{A}} = \infty)\xi_\infty^1 \text{ as } t \to \infty,$$

where $\tau^{\mathbf{A}} = \inf\{t : \xi_t^{\mathbf{A}} = \Phi\}$.

Proof. From Lemma 3, Lemma 5 and Lemma 6, we can easily get Theorem 6.