Contact Process on Trees in Random Environments

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Abstract We study the contact process on T_d in a random environment, in which the recovery rates $\delta(x)$ at site x are *i.i.d.* random variables (bounded above) while the infection rate is a constant λ. We show the two critical points are constants *a.e.*. If $\delta(x)$ satisfies with $\liminf_{v\to\infty} v^\kappa P(-\log\delta(x) > v) > 0$ for some $\kappa > 0$, the contact process locally survives for every $\lambda > 0$.

Key words: Contact process, random environment, Persistent survival

1 Introduction

A number of papers have been written to study the contact process in a random environment, see [1] [2] [4] etc. But all those papers are considered the contact process on lattices. In this paper, we will be primarily concerned the contact process on homogeneous trees in a random environment. We denote a homogeneous tree of degree $d+1$ as T_d. Let o be a distinguished vertex of T_d which we call the root. Contact process on homogeneous trees in a random environment is a continuous time Markov process with state space $\{0,1\}^{T_d}$ with transitions:

$1 \to 0$ at site x with rate $\delta(x)$,

and

$0 \to 1$ at site x with rate $\lambda N(x)$,

where $N(x)$ is the number of infected neighbors of x. We will assume that the local environment $(\delta(x))$, $x \in T_d$ are *i.i.d.* random variables on some probability space $(\Omega, \mathcal{F}, P)$. We shall also assume that the $\delta(x)$'s are strictly positive, and they have a uniform upper bound $\bar{\delta}$. Denote $\delta = \{\delta(x);\ x \in T_d\}$.

We will denote by $(\xi_t^o : t \geq 0)$ the contact process starting from $\xi_0^o = o$.

Let $\tilde{P}$ denote the joint distribution of δ and the $(\lambda, \delta)-$contact process; let $\tilde{P}_\delta$ denote the probability distribution of the $(\lambda, \delta)-$contact process. When $\delta(x) \equiv 1$, we known that the contact process have two critical points $0 < \lambda_1 < \lambda_2 < \infty$. We say the process dies out when $\lambda < \lambda_1$, survives globally but not locally when $\lambda \in (\lambda_1, \lambda_2)$ and survives locally when $\lambda > \lambda_2$. Similar with the definition in the case $\delta(x) \equiv 1$, we define

$$\begin{aligned} \lambda_1(\delta) &= \inf_\lambda \{\tilde{P}_\delta(|\xi_t^o| \geq 1 \text{ for all } t) > 0\}, \\ \lambda_2(\delta) &= \inf_\lambda \{\tilde{P}_\delta(|\xi_t^o(o)| = 1 \ i.o.\ ) > 0\}. \end{aligned}$$

And we have the following theorem.

Theorem 1. *For the contact process in a random environment, we have $\lambda_1(\delta)$ and $\lambda_2(\delta)$ are constants a.s..*

Theorem 2. *For the contact process in a random environment $\{\delta(x),\ x \in T_d\}$, where $\{\delta(x)\}$ are i.i.d.. We have the first critical point of the contact process $\lambda_1 < E\delta(x)/(d-1)$.*

When $\delta(x) \equiv 1$, we have $0 < \lambda_1 < \lambda_2 < \infty$. But when $\delta(x)$ are *i.i.d.* random variables, we have

Theorem 3. *If the recovery rates satisfy with*

$$\liminf_{v \to \infty} v^\kappa P(-\log \delta(x) > v) > 0, \quad \text{for some } \kappa > 0, \tag{1.1}$$

we have that the contact process locally survives as long as $\lambda > 0$, that is $\lambda_1 = \lambda_2 = 0$.

2 Critical Points

Given $\delta \in \Omega$, if $\tilde{P}_\delta(|\xi_t^o| \geq 1 \text{ for all } t) > 0$, we call the process global survival under δ; and if $\tilde{P}_\delta(o \in \xi_t^o \text{ infinitely often }) > 0$, we call the process local survival under δ.

First, we shall show that in a random environment both global survival and local survival have $0-1$ law. We know that T_d is a Cayley graph of a finitely generated group. Suppose that g is a generator of T_d, and write $g(\delta(x)) \doteq \delta(g(x))$, $g(\delta) \doteq \{\delta(g(x)),\ x \in T_d\}$. Then for any $A \in \mathcal{F}$, $P(g^{-1}(A)) = P(A)$. So g is a $P-$preserving transformation.

Let $\tilde{\mathcal{F}} = \{A \in \mathcal{F}; P(g^{-1}A \bigtriangleup A) = 0\}$.

Lemma 2.1. *For any A which is invariant with respect to g, i.e. $P(g^{-1}A \bigtriangleup A) = 0$, we have $P(A) = 0$ or 1.*

Proof: Given $A \in \tilde{\mathcal{F}}$, A is decided by finite vertices in T_d, i.e., $A \in \sigma(\delta(v_1), \cdots, \delta(v_n))$. Then there is a m large enough, which has

$$P(A) = P(g^m A \cap A) = P(g^m A)P(A) = P(A)^2.$$

So $P(A) = 0$ or 1. In the following, we will prove this is true for any $A \in \tilde{\mathcal{F}}$

Given $B \in \tilde{\mathcal{F}}$, B is decided by finite vertices in T_d, we define

$$\mathcal{A}_B = \{A \in \tilde{\mathcal{F}} : \lim_{n\to\infty} P(g^n B \cap A) = P(B)P(A)\}$$

a)Firstly, this is true for any $A \in \tilde{\mathcal{F}}$ which is decided by finite vertices in T_d. So $\mathcal{A}_B \neq \Phi$.

b) Secondly,

1) If $A \in \mathcal{A}_B$

$$\begin{aligned}\lim_{n\to\infty} P(g^n B \cap A^c) &= \lim_{n\to\infty} P(g^n B) - P(g^n B \cap A) \\ &= P(B) - P(B)P(A) = P(B)P(A^c).\end{aligned}$$

So we have $A^c \in \mathcal{A}_B$.

2) If A_1 and $A_2 \in \mathcal{A}_B, A_1 \cap A_2 = \Phi$,

$$\begin{aligned}\lim_{n\to\infty} P(g^n B \cap (A_1 \cup A_2)) &= \lim_{n\to\infty} P((g^n B \cap A_1) \cup (g^n B \cap A_2)) \\ &= \lim_{n\to\infty} P(g^n B \cap A_1) + P(g^n B \cap A_2) \\ &= P(B)(P(A_1) + P(A_2)) \\ &= P(B)P(A_1 \cup A_2).\end{aligned}$$

So $A_1 \cup A_2 \in \mathcal{A}_B$.

3) If $\{A_m\}_{m\geq 1}$ are monotonic increasing in m , $A_m \in \mathcal{A}_B$ and $\lim_{m\to\infty} A_m = A$, then

$$\begin{aligned} & P(g^n B \cap A_m) \leq P(g^n B \cap A) \\ & = P(g^n B \cap A_m) + P(g^n B \cap (A \setminus A_m)) \\ & \leq P(g^n B \cap A_m) + P(A \setminus A_m). \end{aligned} \tag{2.1}$$

For (2.1), we first let $n \to \infty$, then let $m \to \infty$, then

$$\lim_{n\to\infty} P(g^n B \cap A) = P(B)P(A).$$

So $A \in \mathcal{A}_B$.

By 1), 2) and 3), we get $\mathcal{A}_B$ is a $\lambda-$class.

We also know $\mathcal{A}_B$ is a $\pi-$class. So $\mathcal{A}_B$ is a $\sigma-$field. In addition with a), we have

$$\mathcal{A}_B = \tilde{\mathcal{F}}. \tag{2.2}$$

Let $\mathcal{B} = \{B \in \tilde{\mathcal{F}} : \lim_{n\to\infty} P(g^n B \cap A) = P(B)P(A), \forall A \in \tilde{\mathcal{F}}\}$.

a)' First, by (2.2) this is true for $B \in \tilde{\mathcal{F}}$ and B is decided by finite vertices in T_d, so $\mathcal{B} \neq \Phi$.

b)' Secondly,

1)' For any $B \in \mathcal{B}$, we have

$$P(g^n(B^c) \cap A) = P((g^n B)^c \cap A) = P(A) - P(A \cap g^n B).$$

Then

$$\lim_{n\to\infty} P(g^n(B^c) \cap A) = P(A) - P(A)P(B) = P(A)P(B^c).$$

Then $B^c \in \mathcal{B}$.

2)' For any B_1 and $B_2 \in \mathcal{B}, B_1 \cap B_2 = \Phi$,

$$\begin{aligned} \lim_{n\to\infty} P(g^n(B_1 \cup B_2) \cap A) &= \lim_{n\to\infty} P((g^n B_1 \cap A) \cup (g^n B_2 \cap A)) \\ &= \lim_{n\to\infty} P(g^n B_1 \cap A) + P(g^n B_2 \cap A) \\ &= P(A)(P(B_1) + P(B_2)) \\ &= P(A)P(B_1 \cup B_2). \end{aligned}$$

So $B_1 \cup B_2 \in \mathcal{B}$.

3)' If $\{B_m\}_{m\geq 1}$ are monotonic increasing in m , $B_m \in \mathcal{A}_B$ and $\lim_{m\to\infty} B_m = B$, then

$$\begin{aligned} P(g^n B \cap A) &\geq P(g^n(B_m) \cap A) + P(g^n(B \setminus B_m)) \\ &= P(g^n(B_m) \cap A) + P(B \setminus B_m). \end{aligned} \tag{2.3}$$

For (2.3), first we let $n \to \infty$, then $m \to \infty$. We get

$$\lim_{n\to\infty} P(g^n B \cap A) = P(B)P(A).$$

So $B \in \mathcal{B}$.

By 1)', 2)' and 3)' we get $\mathcal{B}$ is a $\lambda-$class.

We also know $\mathcal{B}$ is a $\pi-$class. So $\mathcal{B}$ is a $\sigma-$field. In addition with a)', we get

$$\mathcal{B} = \tilde{\mathcal{F}}.$$

Consequently, if A is invariant with respect to g, we have

$$P(A) = \lim_{n\to\infty} P(g^n A \cap A) = P(A)^2.$$

Then for any A which is invariant with respect to g, we have $P(A) = 0$ or $P(A) = 1$. $\square$

Proof of Theorem 1. Suppose g is a generator of T_d. For any $\delta \in \Omega$, we shall prove

1) If $\tilde{P}_\delta(|\xi_t^o| > 0 \text{ for all } t) > 0$, then $\tilde{P}_{g(\delta)}(|\xi_t^o| > 0 \text{ for all } t) > 0$;

2)If $\tilde{P}_\delta(\xi_t^o(o) = 1,\ i.o.) > 0$, then $\tilde{P}_{g(\delta)}(\xi_t^o(o) = 1,\ i.o.) > 0$.

Clearly, we can see

$$\begin{aligned} \tilde{P}_{g(\delta)}(|\xi_t^o| > 0 \text{ for all } t) &\geq \tilde{P}_{g(\delta)}(|\xi_t^{g(o)}| > 0 \text{ for all } t)\tilde{P}_{g(\delta)}(\xi_1^o(g(o)) = 1) \\ &= \tilde{P}_\delta(|\xi_t^o| > 0 \text{ for all } t)\tilde{P}_{g(\delta)}(\xi_1^o(g(o)) = 1) \\ &> 0. \end{aligned}$$

$$\begin{aligned} \tilde{P}_{g(\delta)}(o \in \xi_t^o\ i.o.) &\geq \tilde{P}_{g(\delta)}(g(o) \in \xi_t^{g(o)}\ i.o.)\tilde{P}_{g(\delta)}(g(o) \in \xi_1^0)\tilde{P}_{g(\delta)}(o \in \xi_1^{g(o)}) \\ &= \tilde{P}_\delta(o \in \xi_t^o\ i.o.)\tilde{P}_{g(\delta)}(o \in \xi_1^{g(o)})\tilde{P}_{g(\delta)}(g(o) \in \xi_1^0) \\ &> 0. \end{aligned}$$

So $\{\delta : \tilde{P}_\delta(|\xi_t^o| \geq 1 \text{ for all } t\ ) > 0\}$ and $\{\delta : \tilde{P}_\delta(o \in \xi_t^o = 1\ i.o.\ ) > 0\}$ are strongly invariant with respect to g. So we have that both $P(\delta : \tilde{P}_\delta(|\xi_t^o| \geq 1 \text{ for all } t) > 0)$ and $P(\delta : \tilde{P}_\delta(|\xi_t^o(o)| = 1\ i.o.\ ) > 0)$ are either 0 or 1. So $\lambda_1(\delta)$ and $\lambda_2(\delta)$ are constants *a.s.*. Denote them as λ_1 and λ_2 respectively. □

Proof of Theorem 2. We just consider the forward contact process, *i.e.*, each vertex in T_d can be infected only once. Given $x \in T_d$, the expected number of children of x that have been infected is $Ed\lambda/(\delta(x) + \lambda)$. So the forward contact process is global survival if and only if $Ed\lambda/(\delta(x)+\lambda) \geq 1$. While $Ed\lambda/(\delta(x)+\lambda) \geq d\lambda/(E\delta(x) + \lambda)$. We have when $\lambda \geq E\delta(x)/(d-1)$, the contact process survives globally, then we have $\lambda_1 < E\delta(x)/(d-1)$. □

3 Persistent Survival

In this section we make some notation first. Given a tree T_d, we immerse the graph Z^+ which has vertices $\{0, 1, 2, \cdots\}$ and edges connecting points which differ by one unit into the tree. The vertex i in Z^+ is an arbitrary neighbor of $i-1$, and different from $\{0, \cdots, i-1\}$. The site 0 of Z^+ is the root of T_d. Removing one of the neighbors of the root, we define the remaining connected component which contains the root as T_d^+. We suppose that the removed vertex is not the vertex 1, so the set of sites $\{0, 1, 2, \cdots\}$ is contained in T_d^+. For any $v \in T_d^+$, if we remove the vertex who is adjacent to v and stays nearer to the root than v, T_d is broken into two parts; we denote the connected part which contains v as $T_d^+(v)$. The distance $|v_1 - v_2|$ between two vertices $v_1, v_2 \in T_d$ is defined to be the number of edges of the shortest path between them. And $|v|$ is the distance between v and the root 0. Let $B^+(x, n) = \{v \in T_d^+ : |v - x| \leq n\}$. To solve the persistent survival problem, we shall first construct a coupling between the contact process and a directed percolation. This coupling based on the graphical construction of the contact process. But the graphical construction we made here is a little different as usual.

For any $x \in T_d$, let

$$L^+(x) = \{(x,s) : s > |x|\} \quad L^-(x) = \{(x,s) : s \le |x|\}.$$

We first generate cuts in $L^-(x)$; these will play no role in our coupling. Then independently of the cuts in L^-, we will generate the cuts in L^+ in a rather special way. For any $t \in \{0\} \cup Z^+$, define

$$I(x,t) = \{x\} \times (|x| + 2t, |x| + 2t + 2],$$

then $L^+(x) = \cup_t I(x,t)$. For each $I(x,t)$, we take(independently for different intervals) independent random variables $H_0(x,t), \eta(x,t), H_1(x,t), H_2(x,t), \cdots$, where $H_0(x,t)$ is exponential with mean $(2\bar{\delta})^{-1}$, $\eta(x,t)$ takes the value 0 and 1 with probabilities $1 - \delta(x)/2\bar{\delta}$ and $\delta(x)/2\bar{\delta}$, and each $H_j(x,t)$ is exponential with mean $\delta(x)$. The cuts within $I(x,t)$ are then located as follows. Define $u_m(x,t) = H_0(x,t) + \cdots + H_m(x,t)$. If $H_0(x,t) > 2$ then there are no cuts within $I(x,t)$. If $H_0(x,t) \le 2$ and $\eta(x,t) = 0$, then the set of cut locations within $I(x,t)$ is in

$$\{(x, x + 2t + u_m(x,t)) : m \ge 1 \text{ and } u_m(x,t) \le 2\}. \tag{3.1}$$

If $H_0(x,t) \le 2$ and $\eta(x,t) = 1$, then in addition to the cuts given by (3.1), there is one more cut located at $(x, x+2t+H_0(x,t))$. Notice that if $\eta(x,t) = 0$ and $H_1(x,t) > 2$, there is no cuts within $I(x,t)$, regardless of the value of $H_0(x,t)$. Now, we define the up edge from (x,t) to $(x,t+1)$ to be open if $\eta(x,t) = 0$ and $H_1(x,t) > 2$; otherwise, it is closed. This up edge from (x,t) is thus open with probability

$$p_x = (1 - \frac{\delta(x)}{2\bar{\delta}}) \exp[-2\delta(x)].$$

Suppose $x, y \in T_d$, $|x - y| = 1$, and $|y| > |x|$. We define the tree edge from (x,t) to (y,t) to be open if $H_0(x,t) > 2$ and there is at least one arrow from the upper half of $I(x,t)$ to the bottom half of $I(y,t)$ (i.e., for some n, the arrow time coordinate is in $(|x| + 2t + 1, |x| + 2t + 2]$); otherwise the tree edge from (x,t) to (y,t) is closed. We define the tree edge from (y,t) to (x,t) to be open if $H_0(x,t) > 2$ and there is at least one arrow from the upper half of $I(y,t)$ to the bottom half of $I(x,t+1)$ (i.e., for some n, the arrow time coordinate is in $(|y|+2t+1, |y|+2t+2])$; otherwise the tree edge from (y,t) to (x,t) is closed. So the double-directed tree edges are open with probability

$$\epsilon = \exp[-2(2\bar{\delta})][1-\exp(-\lambda)],$$

on either direction. Until now, the coupling between the contact process in a random environment and the $(\epsilon, \{p_x\})-$percolation on $T_d \times Z$ has been constructed. Let us denote by P^* the joint distribution of $\{p_x\}$ and the the $(\epsilon, \{p_x\})-$percolation model. For this directed percolation, first, we can see the tree edges are open or closed independent of the vertical edges; secondly, each open edge in the discrete time model implies the existence of a corresponding path for the spread of infection in the original model.

Denote $(0,0) \xrightarrow{G} (m,*)$ the event that there is an open path from $(0,0)$ to (m,n) for some $n>0$ which is contained in the graph G. Let

$$Y_n = P^*((0,0) \overset{B^+(0,n)}{\longrightarrow} (n,*))^{\frac{1}{n}}.$$

By Harris inequality, we have

$$Y_{m+n} > Y_m Y_n.$$

Therefore $\lim_{n\to\infty} Y_n = \alpha$ exists.

Lemma 3.1. *For the $(\epsilon, \{p_x\})-$percolation model, if*

$$\lim_{n\to\infty} Y_n > 1/\sqrt{d}, \tag{3.2}$$

then percolation occurs.

Proof. For (3.2), take n large enough such that $(Y_m)^{\frac{1}{m}} \geq a > \frac{1}{\sqrt{d}}$ for any $m>n$. For $j=1,2,\cdots$, set $\bar{Z}_j = |\{x \in B^+(0,*) : (0,0) \overset{B^+(0,jn)}{\longrightarrow} (x,js)\}|$. Clearly $\bar{Z}_j > Z_j$, where $(Z_j)_{j=0,1,2,\ldots}$ is a branching process with offspring distribution identical to the distribution of $\bar{Z}_1$. The distribution is concentrated on $\{0,1,\cdots d^n\}$ and has mean $\mu = d^n Y_{n,s} \geq (da)^n$. Since the offspring distribution has a finite support and in particular a finite second moment, it follows from the standard branching process theory that for some random variable X with mean $EX = 1$,

$$\frac{Z_j}{(\mu)^j} \longrightarrow X \quad a.s. \text{ as } j \to \infty.$$

In particular, there is $\epsilon > 0$ such that

$$P^*(\bar{Z}_l \geq \frac{(da)^{nl}}{2}) \geq P^*(Z_l \geq \frac{(da)^{nl}}{2}) \geq P^*(Z_l \geq \frac{(\mu)^l}{2}) \geq \epsilon. \tag{3.3}$$

Choose now l large enough for (3.3) to hold, and also that

$$(1 - \frac{\epsilon}{2} a^{nl})^{\frac{(da)^{nl}}{2}} \leq \frac{1}{2}.$$

This last requirement can be fulfilled because

$$\limsup_{l \to \infty} (1 - \frac{\epsilon}{2} a^{nl})^{\frac{(da)^{nl}}{2}} \leq \limsup_{l \to \infty} \exp(-\frac{\epsilon}{4} (da^2)^{nl}) = 0$$

since $ba^2 > 1$.

Denote $L = nl$. Denote $F_i(x)$ the event that is an open path that starts at x exits $B^+(x, L)$ and then return to x. Next we will show inductively in i that

$$P^*(F_i(o)) \geq \frac{\epsilon}{2} a^{nl}. \tag{3.4}$$

For $i = 0$ inequality (3.4) is clearly true. Now we will show that if it is true for i, it is true for $i + 1$. For the spherical symmetry of T_d, we have

$$P^*(F_{i+1}(o)) \geq P^*(\cup_{x \in Z_l} F_i(x) | Z_l \geq \frac{(da)^L}{2}) P^*(Z_l \geq \frac{(da)^L}{2}) Y_L^L, \tag{3.5}$$

by the FKG inequality. We write $F = \cap_{n \geq 1} F_n$. So $P^*(F) \geq \epsilon$. On the event F, there are open paths starting at o wander arbitrarily far away from o and then return to o. So we have percolation occurs. □

Proof of Theorem 3. Consider the percolation constructed above. Let Y_0 be the largest y such that every up edge between $(0, 0)$ and $(0, y)$ is open. Let $D_0^i (i = 1, \cdots, Y_0)$ be the largest d such that every right edge between $(0, i)$ and (d, i) is open and let

$$R_0 = \max(D_0^1, \cdots, D_0^{Y_0}),$$

providing $Y_0 \geq 1$. If $Y_0 = 0$, set $R_0 = 0$.

$$\begin{aligned}
P^*(R \geq r) &= 1 - \sum_{y=0}^{\infty} P^*(Y = y)[P^*(D^1 \leq r)]^y \\
&= 1 - \sum_{y=0}^{\infty} E((1-p_x)p_x)^y (1-\epsilon^r)^y \\
&= 1 - E\frac{1-p_x}{1-p_x(1-\epsilon^r)} \\
&= E\frac{p_x\epsilon^r}{1-p_x+p_x\epsilon^r}
\end{aligned}$$

It's easy to see that

$$P^*(R \geq r+s)^{\frac{1}{r+s}} > (P^*(R \geq r))^{\frac{1}{r}}.$$

So $\lim_{r\to\infty} P^*(R \geq r)^{\frac{1}{r}}$ exists.

So if $\lim_{r\to\infty}(P^*(R \geq r))^{\frac{1}{r}} > \frac{1}{\sqrt{d}}$, we have percolation occurs. Setting $V = -\log(1-p_x)$, $c = -\log\epsilon$, then

$$\begin{aligned}
P^*(R \geq r) &= E\tfrac{p_x\epsilon^r}{1-p_x+p_x\epsilon^r} \\
&= E\tfrac{1-e^{-V}}{1-e^{-V}+e^{-V+cr}} \\
&= \textstyle\int_0^\infty e^{v-cr}[e^{v-cr} - e^{-cr} + 1]^{-2} P(V > v)dv.
\end{aligned} \tag{3.6}$$

However, $p_x \geq (1 - \delta(x)/2\bar{\delta})(1 - 2\delta(x)) \geq 1 - b\delta(x)$, where $b = (2\bar{\delta})^{-1} + 2$, so that

$$-\log(1-p_x) \geq -\log\delta(x) - b'. \tag{3.7}$$

Now by (3.7), if (1.1) holds, then there exists $0 < v_0 < \infty$, which have

$$P(V > v) \geq \frac{\Gamma}{v^\kappa},$$

for some $\Gamma > 0$ and $\kappa > 0$. Inserting this into (3.6), letting $v' = v - cr$ and taking a limit yields

$$\begin{aligned}
\liminf_{r\to\infty} r^\kappa P^*(R \geq r) &= \liminf_{r\to\infty} \int_{v_0-cr}^{\infty} e^v[e^v - e^{-cr} + 1]^{-2} \frac{\Gamma r^\kappa}{(v+cr)^\kappa} dv \\
&\geq \frac{\Gamma}{c^\kappa} \int_{-\infty}^{\infty} e^v[e^v+1]^{-2} dv \\
&= \frac{\Gamma}{c^\kappa}.
\end{aligned}$$

So

$$\lim_{r\to\infty}(P^*(R\geq r))^{\frac{1}{r}}=1>\frac{1}{\sqrt{d}}.$$

It is clear that $P^*(R\geq r)\leq P^*((0,0)\to(r,*))$. So we get percolation occurs on the $(\epsilon,p_x)-$percolation model and there is an infinite path starting from o wandering arbitrarily far away from o and then return to o. By the coupling we constructed at the beginning of this section. We know that this means there are infection paths starting at o that wander arbitrarily far away from e and then return to e. But a contact process initiated by a single infected site o can only reach finitely many sites in finite time. Consequently, we have the root is reinfected at arbitrarily large times, and

$$P^*(\xi_t^o(o)=1,\ i.o.)>0$$

so we get the contact process locally survives for any $\epsilon>0$, and $\lambda_1=\lambda_2=0$. $\square$

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